Analysis of the Rosenblatt process

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Abstract

We analyze *the Rosenblatt process* which is a selfsimilar process with stationary increments and which appears as limit in the so-called *Non Central Limit Theorem* (Dobrushin and Major (1979), Taqqu (1979)). This process is non-Gaussian and it lives in the second Wiener chaos. We give its representation as a Wiener-Itô multiple integral with respect to the Brownian motion on a finite interval and we develop a stochastic calculus with respect to it by using both pathwise type calculus and Malliavin calculus.

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1 Introduction

A selfsimilar object is exactly of approximately similar to a part of itself. Selfsimilar processes are invariant in distribution under suitable scaling. They are of considerable interest in practice since aspects of the selfsimilarity appear in different phenomena like telecommunications, economics, hydrology or turbulence. We refer to the work of Taqqu [43] for a guide on the appearance of the selfsimilarity in many applications and to the monographs by Samorodnitsky and Taqqu [39] and by Embrechts and Maejima [13] for complete expositions on selfsimilar processes.

In this work we analyze a special class of selfsimilar processes that are limits in the so called *Non Central Limit Theorem* (see Dobrushin and Major [11] or Taqqu [42]). Let us briefly recall the general context.

Consider $(\xi_n)_{n \in \mathbb{Z}}$ a stationary Gaussian sequence with mean zero and variance 1 such that its correlation function satisfies

$$r(n) := \mathbf{E}\left(\xi_0 \xi_n\right) = n^{\frac{2H-2}{k}} L(n) \tag{1}$$

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with $H \in (\frac{1}{2}, 1)$ and L is a slowly varying function at infinity (see e.g. [13]). Denote by $H_m(x)$ the Hermite polynomial of degree m given by $H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}}$. Let g be a function such that $\mathbf{E}(g(\xi_0)) = 0$ and $\mathbf{E}(g(\xi_0^2)) < \infty$. Suppose that g *has Hermite rank* equal to k; that is, if g admits the following expansion in Hermite polynomials

$$g(x) = \sum_{j \geq 0} c_j H_j(x), \quad c_j = \frac{1}{j!} \mathbf{E}\left(g(\xi_0 H_j(\xi_0))\right)$$

then

$$k = \min\{j; c_j \neq 0\}.$$

Since $\mathbf{E}[g(\xi_0)] = 0$, we have $k \geq 1$. Then the Non Central Limit Theorem ([11], [42]) says that

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\xi_j)$$

converges as $n \to \infty$ in the sense of finite dimensional distributions to the process

$$Z_H^k(t) = c(H,k) \int_{\mathbb{R}^k} \int_0^t \left(\prod_{j=1}^k (s - y_i)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} \right) ds dB(y_1) \dots dB(y_k), \qquad (2)$$

where $x_+ = \max(x, 0)$ and the above integral is a multiple Wiener-Itô stochastic integral with respect to a Brownian motion $B(y))_{y \in \mathbb{R}}$ (see [27] for the definition). The constant $c(H, k)$ is positive and it will be taken such that $\mathbf{E}\left(Z_H^k(1)^2\right) = 1$. The process $(Z_H^k(t))_{t \geq 0}$ is called *the Hermite process* and it is H-selfsimilar in the sense that for any $c > 0$, $(Z_H^k(ct)) =^{(d)} (c^H Z_H^k(t))$, where " $=^{(d)}$ " means equivalence of all finite dimensional distributions, and it has stationary increments.

When $k = 1$ the process given by (2) is nothing else that the *fractional Brownian motion (fBm)* with Hurst parameter $H \in (\frac{1}{2}, 1)$. For $k \geq 2$ the process is not Gaussian. If $k = 2$ then the process (2) is known as *the Rosenblatt process* (it has actually called in this way by M. Taqqu in [41]).

The fractional Brownian motion is of course the most studied process in the class of Hermite processes due to its significant importance in modeling. A stochastic calculus with respect to it has been intensively developed in the last decade. We refer, among others, to [4], [5], [9], [16].

Our main interest consists here in the study, from the stochastic calculus point of view, of the Rosenblatt process. Although it received a less important attention than the fractional Brownian motion, this process is still of interest in practical applications because of its self-similarity, stationarity of increments and long-range dependence. There exists a consistent literature that focuses on different theoretical aspects of the Rosenblatt processes. Let us recall some of these works. For example, extremal properties of the Rosenblatt distribution have been studied by J.M. Albin in [2] and [3]. The rate of convergence to the Rosenblatt

process in the Non Central Limit Theorem has been given by Leonenko and Ahn [23]. Pipiras [29] and Pipiras and Abry [30] studied the wavelet-type expansion of the Rosenblatt process. A law of iterated logarithm has been given is [15].

Among the applications of the Rosenblatt process in statistics or econometrics, we mention the following.

- In the unit root testing problem with errors being nonlinear transforms of linear processes with long-range dependence, the asymptotic distributions in the model are shown in [46] to be functionals of Hermite processes.

- limiting distributions of the parabolically rescaled solutions of the heat equation with singular non-Gaussian data have similar behavior to the Rosenblatt distribution (see [24])

- the Rosenblatt distribution also appears to be the asymptotic distribution of an estimator related to the semiparametric bootstrap approach to hypothesis tests (see [18]) or to the estimation of the long-range dependence parameter ([21])

Besides these more or less practical applications of the Rosenblatt process, denoted in the following by Z, our motivation is also theoretical; it comes from the recent intensive interest to push further the stochastic calculus with respect to more and more general integrator processes. We believe that this process constitutes an interesting and instructive example where the recent developed techniques of the generalized stochastic calculus can find a significant test bench.

We actually use the two principal methods to develop a stochastic integration theory: the pathwise type calculus and the Malliavin calculus/Skorohod integration. The first approach (that includes essentially the rough paths analysis, see [34], and the stochastic calculus via regularization, see [36]) can be directly applied to the Rosenblatt process because of its regular paths and of the nice covariance structure; a pathwise Itô formula can be written and Stratonovich stochastic equation with Z as noise can be considered. The Malliavin calculus and the Skorohod integration are in general connected in a deeper way to the Gaussian structure of the integrator process and as it will be seen in the present work, Skorohod Itô formula can be derived only in particular cases. Although the formula we obtain is rather complicated and not easily tractable, the principal signification of the result is the fact that one can precisely see here that the Gaussian nature of the integrator process is decisive in the stochastic integration theory; once we go out from the Gaussian context, one cannot obtain Itô's formulas that end by a second derivative term.

We organized our paper as follows. Section 2 presents basic properties of the Rosenblatt process. In particular we prove a stochastic integral representation on a finite integral that will be useful for the construction of the stochastic calculus. In Section 3 we introduce Wiener integrals with respect to Z by following the ideas in [25] and [22]. We define in Section 4 the Hilbert-valued Rosenblatt process and we consider stochastic evolution equations with this process as noise. Section 5 describes the application of the stochastic calculus via regularization introduced by F. Russo and P. Vallois in [36] to the Rosenblatt process and in Section 6 we

discuss the Skorohod (divergence) integral: we define the integral and we give conditions that ensure the integrability and the continuity of the indefinite integral process. In Section 7 we prove the relation between the pathwise and the divergence integrals: here the pathwise integral is equal to the Skorohod integral plus two trace terms (in the fBm case there is only a trace term). Finally Section 8 contains a discussion on the Itô formula in the Skorohod sense.

2 On the Rosenblatt process

In this section we will analysis some basic properties of the Rosenblatt process; in particular we are interested in its representation as a stochastic integral on a finite interval. As we said, this process is obtained by taking $k = 2$ in the relation (2), so

$$Z_2(t) := Z(t) = a(H) \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_0^t (s - y_1)_+^{-\frac{2-H}{2}} (s - y_2)_+^{-\frac{2-H}{2}} ds \right) dB(y_1) dB(y_2) \qquad (3)$$

where $(B(y), y \in \mathbb{R})$ is a standard Brownian motion on $\mathbb{R}$. The constant $a(H)$ is a positive normalizing constant and it is chosen such that $\mathbf{E}(Z(1)^2) = 1$. It follows actually from [25] that

$$a(H)^2 = \left(\frac{\beta(\frac{H}{2}, H - 1)^2}{2H(2H - 1)} \right)^{-1}$$

Recall that the process $(Z(t))_{t \in [0,T]}$ is selfsimilar of order H and it has stationary increments; it admits a Hölder continuous version of order $\delta < H$. Since $H \in (\frac{1}{2}, 1)$, it follows that the process Z exhibits long-range dependence.

Since our main interest consists in the construction of the stochastic calculus with respect to the process Z, the representation (3) is not very convenient; as in the fBm case, we would like to represent Z_t as a stochastic integral with respect to a Brownian motion with time interval $[0, T]$. Recall that the fBm with $H > \frac{1}{2}$ can be written as

$$B_t^H = \int_0^t K^H(t, s) dW_s \qquad (4)$$

with $(W_t, t \in [0, T])$ a standard Wiener process and

$$K^H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du \qquad (5)$$

where $t > s$ and

$$c_H = \left(\frac{H(2H - 1)}{\beta(2 - 2H, H - \frac{1}{2})} \right)^{\frac{1}{2}}. \qquad (6)$$

Note that to prove the representation (4) (at least in law) it suffices to see that the right member has the same covariance R as the fBm; otherwise, it can be easily seen from the expression of

the kernel K that the right member in (4) is H-selfsimilar with stationary increments and as a consequence it cannot be nothing else but a fractional Brownian motion with parameter H.

Since the Rosenblatt process is not Gaussian, the proof in its case of a similar representation to (4) needs a supplementary argument; in fact we have the following

Proposition 1 *Let K be the kernels (5) and let $(Z(t))_{t\in[0,T]}$ be a Rosenblatt process with parameter H. Then it holds that*

$$Z(t) =^{(d)} d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2) \qquad (7)$$

where $(B_t, t \in [0,T])$ is a Brownian motion,

$$H' = \frac{H+1}{2} \qquad (8)$$

and

$$d(H) = \frac{1}{H+1} \left(\frac{H}{2(2H-1)} \right)^{-\frac{1}{2}}. \qquad (9)$$

Remark 1 *i) The constant $d(H)$ is a normalizing constant, it has been chosen such that $\mathbf{E}(Z(t)Z(s)) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t-s|^{2H} \right)$. Indeed,*

$$
\begin{aligned}
\mathbf{E}(Z(t)Z(s)) &= 2d(H)^2 \int_0^{t\wedge s} \int_0^{t\wedge s} dy_1 dy_2 \\
&\quad \times \left(\int_{y_1\vee y_2}^t \int_{y_1\vee y_2}^s \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) \frac{\partial K^{H'}}{\partial u}(v, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) du dv \right) \\
&= 2d(H)^2 \int_0^t \int_0^s dv du \left(\int_0^{u\wedge v} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(v, y_1) dy_1 \right)^2 \\
&= 2d(H)^2 (H'(2H'-1))^2 \int_0^t \int_0^s |u-v|^{2H-2} dv du = R(t,s).
\end{aligned}
$$

ii) It can be seen without without difficulty that the process

$$Z'(t) := d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2)$$

defines a H selsimilar process with stationary increments. Indeed, for any $c > 0$,

$$Z'(ct) = \int_0^{ct} \int_0^{ct} \left[\int_{y_1 \vee y_2}^{ct} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2)$$

$$= \int_0^{ct} \int_0^{ct} \left[\int_{\frac{y_1}{c} \vee \frac{y_2}{c}}^{t} \frac{\partial K^{H'}}{\partial u}(cu, y_1) \frac{\partial K^{H'}}{\partial u}(cu, y_2) cdu \right] dB(y_1) dB(y_2)$$

$$= \int_0^{t} \int_0^{t} \left[\int_{y_1 \vee y_2}^{t} \frac{\partial K^{H'}}{\partial u}(cu, cy_1) \frac{\partial K^{H'}}{\partial u}(cu, cy_2) cdu \right] dB(cy_1) dB(cy_2)$$

and since $B(cy) =^{(d)} c^{\frac{1}{2}} B(y)$ *and* $\frac{\partial K^{H'}}{\partial u}(cu, cy_i) = c^{H' - \frac{3}{2}} \frac{\partial K^{H'}}{\partial u}(u, y_i)$ *we obtain* $Z(ct) =^{(d)}$ $c^H Z(t)$.

The fact that Z' has stationary increments follows from the relation

$$K^{H'}(t + h, s) - K^{H'}(t, s) = K^{H'}(t - s, h)$$

for any $s, t \in [0, T]$, $s < t$ and $h > 0$.

Proof of Proposition 1: Let us denote by $Z'(t)$ the right hand side of (7). Consider $b_1, \ldots, b_n \in \mathbb{R}$ and $t_1, \ldots, t_n \in [0, T]$. We need to show that the random variables

$$\sum_{l=1}^{n} b_l Z(t_l), \quad \sum_{l=1}^{n} b_l Z'(t_l)$$

have the same distribution.

We will use the following criterium by Fox and Taqqu (see [14]): If $f \in L^2([0, T]^2)$ is a symmetric function, then the law of the multiple Wiener-Itô integral $I_2(f)$ is uniquely determined by its *cumulants*, where the mth cumulant of f is given by

$$c_m(f) = \frac{(m-1)!}{2} 2^m \int_{\mathbb{R}^m} f(x_1, x_2) f(x_2, x_3) \ldots f(x_{m-1}, x_m) f(x_m, x_1) dx_1 \ldots dx_m. \qquad (10)$$

In other words, if two symmetric functions $f, g \in L^2([0, T]^2)$ have the same cumulants, then the multiple Wiener-Itô integrals of order two $I_2(f)$ and $I_2(g)$ have the same law.

We will show that, for every $t, s \in [0, T]$, the random variables $Z_t + Z_s$ and $Z'_t + Z'_s$ have the same law; the general case case will follow by a similar calculation. It holds that

$$Z'_t + Z'_s = I_2(f_{t,s})$$

where

$$f_{t,s}(y_1, y_2) = 1_{[0,t]}(y_1) 1_{[0,t]}(y_2) \int_{y_1 \vee y_2}^{t} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du$$

$$+ 1_{[0,s]}(y_1) 1_{[0,s]}(y_2) \int_{y_1 \vee y_2}^{s} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) duv \qquad (11)$$

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We have denoting by $a_m := \frac{(m-1)!}{2} 2^m d(H)^m$,

$$c_m(f_{s,t})$$
$$= a(m) \int_{\mathbb{R}^m} f_{t,s}(y_1, y_2) \dots f_{t,s}(y_m, y_1) dy_1 \dots dy_m$$
$$= a(m) \int_{\mathbb{R}^m} dy_1 \dots dy_m$$
$$\times \left(\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u_1, y_1) \frac{\partial K^{H'}}{\partial u}(u_1, y_2) du_1 + \int_{y_1 \vee y_2}^s \frac{\partial K^{H'}}{\partial u}(u_1, y_1) \frac{\partial K^{H'}}{\partial u}(u_1, y_2) du_1 \right)$$
$$\times \left(\int_{y_2 \vee y_3}^t \frac{\partial K^{H'}}{\partial u}(u_2, y_2) \frac{\partial K^{H'}}{\partial u}(u_2, y_3) du_2 + \int_{y_2 \vee y_3}^s \frac{\partial K^{H'}}{\partial u}(u_2, y_2) \frac{\partial K^{H'}}{\partial u}(u_2, y_3) du_2 \right)$$
$$\times \dots$$
$$\times \left(\int_{y_m \vee y_1}^t \frac{\partial K^{H'}}{\partial u}(u_m, y_m) \frac{\partial K^{H'}}{\partial u}(u_m, y_1) du_m + \int_{y_m \vee y_1}^s \frac{\partial K^{H'}}{\partial u}(u_m, y_1) \frac{\partial K^{H'}}{\partial u}(u_m, y_m) du_m \right)$$

and by classical Fubini theorem

$$c_m(f_{s,t})$$
$$= a(m) \sum_{t_j \in \{t,s\}} \int_0^{t_1} \dots \int_0^{t_m} du_1 \dots du_m$$
$$\times \left(\int_0^{u_1 \wedge u_m} \frac{\partial K^{H'}}{\partial u_1}(u_1, y_1) \frac{\partial K^{H'}}{\partial u_m}(u_m, y_1) dy_1 \right)$$
$$\times \left(\int_0^{u_1 \wedge u_2} \frac{\partial K^{H'}}{\partial u_1}(u_1, y_2) \frac{\partial K^{H'}}{\partial u_2}(u_2, y_2) dy_2 \right)$$
$$\dots$$
$$\times \int_0^{u_{m-1} \wedge u_m} \frac{\partial K^{H'}}{\partial u_{m-1}}(u_m, y_m) \frac{\partial K^{H'}}{\partial u_m}(u_m, y_m) dy_m$$
$$= a(m) \sum_{t_j \in \{t,s\}} \int_0^{t_1} \dots \int_0^{t_m} du_1 \dots du_m$$
$$|u_1 - u_2|^{2H'-2} |u_2 - u_3|^{2H'-2} \dots |u_m - u_1|^{2H'-2}. \tag{12}$$

with $a(m) = a(m) \left(H'(2H'-1) \right)^m$.

The computation of the cumulant of $Z_t + Z_s$ is similar. Indeed, we can write, for $s, t \in [0, T]$,

$$Z(t) + Z(s) = I_2(g_{s,t})$$

where

$$g_{s,t} = a(H) \left(\int_0^t (u - y_1)_+^{\frac{H-2}{2}} (u - y_2)_+^{\frac{H-2}{2}} du + \int_0^s (u - y_1)_+^{\frac{H-2}{2}} (u - y_2)_+^{\frac{H-2}{2}} du \right)$$

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and the mth cumulant of the kernel $g_{s,t}$ is given by

$$
\begin{aligned}
c_m(g_{s,t}) &= b(m) \int_{\mathbb{R}^m} dy_1 \ldots dy_m \\
&\quad \left(\int_0^t (u_1 - y_1)_+^{\frac{H-2}{2}} (u_1 - y_2)_+^{\frac{H-2}{2}} du_1 + \int_0^s (u_1 - y_1)_+^{\frac{H-2}{2}} (u_1 - y_2)_+^{\frac{H-2}{2}} du_1 \right) \\
&\quad \left(\int_0^t (u_2 - y_2)_+^{\frac{H-2}{2}} (u_2 - y_3)_+^{\frac{H-2}{2}} du_2 + \int_0^s (u_2 - y_2)_+^{\frac{H-2}{2}} (u_2 - y_3)_+^{\frac{H-2}{2}} du_2 \right) \\
&\quad \ldots \\
&\quad \left(\int_0^t (u_m - y_m)_+^{\frac{H-2}{2}} (u_m - y_1)_+^{\frac{H-2}{2}} du_1 + \int_0^s (u_m - y_m)_+^{\frac{H-2}{2}} (u_m - y_1)_+^{\frac{H-2}{2}} du_m \right) \\
&= b(m) \sum_{t_j \in \{t,s\}} \int_0^{t_1} \ldots \int_0^{t_m} du_1 \ldots du_m \\
&= \int_{\mathbb{R}} (u_1 - y_1)_+^{\frac{H-2}{2}} (u_m - y_1)_+^{\frac{H-2}{2}} dy_1 \int_{\mathbb{R}} (u_1 - y_2)_+^{\frac{H-2}{2}} (u_2 - y_2)_+^{\frac{H-2}{2}} dy_2 \\
&\quad \ldots \ldots \int_{\mathbb{R}} (u_{m-1} - y_m)_+^{\frac{H-2}{2}} (u_m - y_m)_+^{\frac{H-2}{2}} dy_m.
\end{aligned}
$$

Since for any $a > 0$

$$
\int_{\mathbb{R}} (u - y)_+^{a-1} (v - y)_+^{a-1} dy = \beta(a, 2a-1) |u - v|^{2a-1}
$$

we get

$$
\begin{aligned}
c_m(g_{s,t}) &= b(m) \beta(\frac{H}{2}, H-1)^m \sum_{t_j \in \{t,s\}} \int_0^{t_1} \ldots \int_0^{t_m} du_1 \ldots du_m \\
&\quad |u_1 - u_2|^{2H'-2} |u_2 - u_3|^{2H'-2} \ldots |u_m - u_1|^{2H'-2}
\end{aligned}
\tag{13}
$$

and it remains to observe that $a'(m) = b(m)$ which implies that (12) equals (13). ∎

From now on we will use the version of the Rosenblatt process given by the right side of (7).

We will finish this section by proving that the Rosenblatt process possesses a similar property to the fBm, that is, it can be approximated by a sequence of semimartingales (here actually, since $H > \frac{1}{2}$, by a sequence of bounded variation processes). In the fBm case, the property is inherited by the divergence integral (see [4], [7], [6]); this fact can be used to construct financial models with the Rosenblatt process as noise (see [6]).

The basic observation is that, if one interchanges formally the stochastic and Lebesque integrals in (7), one gets

$$
Z(t)" = " \int_0^t \left(\int_0^u \int_0^u \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) dB(y_1) dB(y_2) \right) du
$$

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but the above expression cannot hold because the kernel $\frac{\partial K^{H'}}{\partial u}(u, y_1)\frac{\partial K^{H'}}{\partial u}(u, y_2)$ does not belong to $L^2([0, T]^2)$ since the partial derivative $\frac{\partial K^{H'}}{\partial u}(u, y_1)$ behaves on the diagonal as $(u - y_1)^{\frac{H-2}{2}}$.

Let us define, for every $\varepsilon > 0$,

$$
\begin{aligned}
Z^\varepsilon(t) &= d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_1)\frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_2)du\right] dB(y_1)dB(y_2) \\
&= \int_0^t \left(\int_0^u \int_0^u \frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_1)\frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_2)dB(y_1)dB(y_2)\right) du \\
&:= \int_0^t A_\varepsilon(u)du.
\end{aligned}
$$

Since $A_\varepsilon \in L^2([0, T] \times \Omega)$ for every $\varepsilon > 0$ and it is adapted, it follows that the process Z^ε is a semimartingale.

Proposition 2 *For every $t \in [0, T]$, $Z^\varepsilon(t) \to Z(t)$ in $L^2(\Omega)$.*

Proof: We have

$$
\begin{aligned}
Z^\varepsilon(t) - Z(t) &= \int_0^t \int_0^t dB(y_1)dB(y_2) \\
&\left(\int_{y_1 \vee y_2}^t \left(\frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_1)\frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_2) - \frac{\partial K^{H'}}{\partial u}(u, y_1)\frac{\partial K^{H'}}{\partial u}(u, y_2)\right) du\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{E}\,|Z^\varepsilon(t) - Z(t)|^2 &= 2 \int_0^t \int_0^t dy_1 dy_2 \int_{y_1 \vee y_2}^t \int_{y_1 \vee y_2}^t dvdu \\
&\left(\frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_1)\frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_2) - \frac{\partial K^{H'}}{\partial u}(u, y_1)\frac{\partial K^{H'}}{\partial u}(u, y_2)\right) \\
&\left(\frac{\partial K^{H'}}{\partial v}(v + \varepsilon, y_1)\frac{\partial K^{H'}}{\partial v}(v + \varepsilon, y_2) - \frac{\partial K^{H'}}{\partial v}(v, y_1)\frac{\partial K^{H'}}{\partial v}(v, y_2)\right)
\end{aligned}
$$

Clearly the quantity $\left(\frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_1)\frac{\partial K^{H'}}{\partial u}(u + \varepsilon, y_2) - \frac{\partial K^{H'}}{\partial u}(u, y_1)\frac{\partial K^{H'}}{\partial u}(u, y_2)\right)$ converges to zero as $\varepsilon \to 0$ for every u, y_1, y_2 and the conclusion follows by the dominated convergence theorem. ∎

3 Wiener integrals

The covariance structure of the Rosenblatt process allows to construct Wiener integrals with respect to it. We refer to Maejima and Tudor [25] for the definition of Wiener integrals with respect to general Hermite processes and to Kruk and al. [22] for a more general context. Let us recall the main points and translate this construction in our context.

One note that

$$
Z(t) = \int_0^T \int_0^T I\left(1_{[0,t]}\right)(y_1, y_2) dB(y_1) dB(y_2)
$$

where the operator I is defined on the set of functions $f : [0,T] \to \mathbb{R}$ and takes values in the set of functions $g : [0,T]^2 \to \mathbb{R}^2$ and it is given by

$$
I(f)(y_1, y_2) = d(H) \int_{y_1 \vee y_2}^T f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du. \tag{14}
$$

If f is an element of the set $\mathcal{E}$ of step functions on $[0,T]$ of the form

$$
f = \sum_{i=0}^{n-1} a_i 1_{(t_i, t_{i+1}]}, \quad t_i \in [0,T] \tag{15}
$$

then we naturally define its Wiener integral with respect to Z as

$$
\int_0^T f(u) dZ(u) := \sum_{i=0}^{n-1} a_i \left(Z_{t_{i+1}} - Z_{t_i}\right) = \int_0^T \int_0^T I(f)(y_1, y_2) dB(y_1) dB(y_2). \tag{16}
$$

Let $\mathcal{H}$ be the set of functions f such that

$$
\|f\|_{\mathcal{H}}^2 := 2 \int_0^T \int_0^T I(f)(y_1, y_2)^2 dy_1 dy_2 < \infty. \tag{17}
$$

It can be seen that

$$
\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= 2d(H)^2 \int_0^T \int_0^T \left(\int_{y_1 \vee y_2}^T f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right)^2 dy_1 dy_2 \\
&= 2d(H)^2 \int_0^T \int_0^T dy_1 dy_2 \int_{y_1 \vee y_2}^T \int_{y_1 \vee y_2}^T dv du \\
&\quad f(u) f(v) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) \frac{\partial K^{H'}}{\partial v}(v, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) \\
&= 2d(H)^2 \int_0^T \int_0^T \left(\int_0^{u \wedge v} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_1) dy_1 \right)^2 dv du \\
&= H(2H-1) \int_0^T \int_0^T f(u) f(v) |u - v|^{2H-2} dv du.
\end{aligned}
$$

It can be proved as in [25] or [22] that the mapping

$$f \rightarrow \int_0^T f(u)dZ(u)$$

defines an isometry from $\mathcal{E}$ to $L^2(\Omega)$ and it can be extended by continuity to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ because $\mathcal{E}$ is dense in $\mathcal{H}$ (see [32]). We will call this extension the Wiener integral of $f \in \mathcal{H}$ with respect to Z.

Remark 2 *It follows from Pipiras and Taqqu (see [32]) that the space $\mathcal{H}$ contains not only functions but its elements could be also distributions. Therefore it is suitable to know subspaces of $\mathcal{H}$ that are spaces of functions. A such subspace is $|\mathcal{H}|$ where*

$$|\mathcal{H}| = \{f : [0, T] \rightarrow \mathbb{R}| \int_0^T \int_0^T |f(u)||f(v)||u - v|^{2H-2}dvdu < \infty\}.$$

It actually holds

$$L^{\frac{1}{H}}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}.$$

The space $|\mathcal{H}|$ (and hence $\mathcal{H}$) is not complete with respect to the norm $\|\cdot\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|f\|_{|\mathcal{H}|}^2 = H(2H - 1) \int_0^T \int_0^T |f(u)||f(v)||u - v|^{2H-2}dvdu.$$

The Wiener integrals $\int_0^T f(u)dZ(u)$ and $\int_0^T g(u)dZ(u)$ are not necessarily independent when the functions f and g are orthogonal in $\mathcal{H}$. A characterization of their independence is given in the next result.

Proposition 3 *Let $f, g \in \mathcal{H}$. Then $\int_0^T f(u)dZ(u)$ and $\int_0^T g(u)dZ(u)$ are independent if and only if*

$$\langle f(\cdot)\frac{\partial K^{H'}}{\partial u}(\cdot, y_1), g(\cdot)\frac{\partial K^{H'}}{\partial u}(\cdot, y_2)\rangle_{\mathcal{H'}} = 0 \quad a.e.(y_1, y_2) \in [0, T]^2 \tag{18}$$

where $\mathcal{H'}$ is the space analogous to $\mathcal{H}$ corresponding to the Hurst parameter H'.

Proof: We use a result by Üstunel-Zakai [40] (see also Kallenberg [20]): two multiple Wiener-Itô integrals with respect to the standard Wiener process $I_n(f)$ and $I_m(g)$ with f, g symmetric, $f \in L^2[0, T]^n$ and $g \in L^2[0, T]^m$ are independent if and only if $f \otimes_1 g = 0$ a.e. on $[0, T]^{m+n-2}$, where

$$(f \otimes_1 g)(t_1, \ldots, t_{n-1}, s_1, \ldots, s_{m-1}) = \int_0^T f(t_1, \ldots, t_{n-1}, t)g(s_1, \ldots, s_{n-1}, t)dt.$$

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Let us apply the above result to

$$F(y_1, y_2) = 1_{[0,t]^2}(y_1, y_2) \int_{y_1 \vee y_2}^{T} f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du$$

and

$$G(y_1, y_2) = 1_{[0,t]^2}(y_1, y_2) \int_{y_1 \vee y_2}^{T} g(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

Then

$$
\begin{aligned}
(F \otimes_1 G)(y_1, y_2) &= \int_0^T ds \\
&\times \int_{y_1 \vee s}^{T} f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, s) du \int_{y_2 \vee s}^{T} g(v) \frac{\partial K^{H'}}{\partial v}(v, y_2) \frac{\partial K^{H'}}{\partial v}(v, s) dv \\
&= c(H) \int_0^T \int_0^T f(u) g(v) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) |u - v|^{2H'-2} dv du
\end{aligned}
$$

and the conclusion follows easily. ∎

As an immediate consequence, we obtain

Corollary 1 *If $f \otimes g = 0$ a.e. on $[0,T]^2$ then the random variables $\int_0^T f(u) dZ(u)$ and $\int_0^T g(u) dZ(u)$ are independent.*

Remark 3 *The construction of Wiener integrals with respect to the Rosenblatt process allows to consider associated Ornstein-Uhlenbeck processes. This has been done in [25] for general Hermite processes of order k following the argument in [8] for the fBm case. It can be showed that the equation*

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma Z(t), \quad t \geq 0, \tag{19}$$

where $\sigma, \lambda > 0$ and the initial condition ξ is a random variable in $L^0(\Omega)$ has an unique solution that can be represented as

$$X^\xi(t) = e^{-\lambda t} \left(\xi + \sigma \int_0^t e^{\lambda u} dZ(u) \right), \quad t \geq 0.$$

where the stochastic integral above exists in the Wiener sense. When the initial condition is $\xi = \sigma \int_{-\infty}^0 e^{\lambda u} dZ(u)$ (in [8] the integrals are considered on the whole real line), the solution of (19) can be written as

$$X(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dZ(u) \tag{20}$$

and it is called the stationary Rosenblatt Ornstein-Uhlenbeck process.

Remark 4 *The Non Central Limit Theorem given by [11], [42] can be extended to Wiener integrals (see [25]). More precisely, under suitable assumptions on the deterministic function $f \in \mathcal{H}$ one obtains that the sequence*

$$\frac{1}{n^H} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) g(\xi_j)$$

converges weakly when $n \to \infty$, to the Wiener integral $\int f(u) dZ(u)$ (g and ξ_j were introduced in Section 1).

4 Infinite dimensional process and stochastic evolution equations

In this part we define a Hilbert-valued Rosenblatt process and we consider stochastic evolution equations driven by it.

Let us consider U a real and separable Hilbert space and Q a nuclear, self-adjoint positive and nuclear operator on U. There exists then a sequence $0 < \lambda_n \searrow 0$ of eigenvalues of Q such that $\sum_{n \geq 1} \lambda_n < \infty$. Moreover the corresponding eigenvectors form an orthonormal basis in U. We define the *infinite dimensional Rosenblatt process on U* as

$$Z(t) = \sum_{\nu \geq 0} \sqrt{\lambda_n} e_n z_j(t) \tag{21}$$

where $(z_j)_{j \geq 0}$ is a family of real independent Rosenblatt processes.

Note that the series (21) is convergent in $L^2(\Omega)$ for every $t \in [0, T]$ since

$$\mathbf{E}\left|Z(t)\right|^2 = \sum_{j \geq 1} \lambda_j \mathbf{E}(z_j^2) = t^{2H} \sum_{j \geq 1} \lambda_j < \infty.$$

Note also that *Z has covariance function $R(t, s)$* in the sense that for every $u, v \in U$, and for every $s, t \in [0, T]$

$$\mathbf{E}\langle Z(t), u \rangle_U \langle Z(s), v \rangle_U = R(t, s) \langle Qu, v \rangle_U.$$

This can be proved exactly as fBm case (see [45]).

In some situations the assumption that Q is nuclear is not convenient. For example one cannot take Q to be the identity operator, that is $\lambda_n = 1$ for every n. Therefore, if $\sum_n \lambda_n = \infty$ we will consider a bigger real and separable Hilbert space $U_1 \supset U$ such that the inclusion $U \subset U_1$ is nuclear. Then the quantity

$$Z(t) = \sum_j z_j(t) e_j \tag{22}$$

is well-defined stochastic process in U_1.

In the sequel we will consider the infinite dimensional Rosenblatt process to be defined by (22).

Following the one dimensional case, one can introduce Wiener integrals with respect to the Hilbert-valued process Z. Let V be another Hilbert space Let $(\Phi_s, s \in [0,T])$ a stochastic process with valued in the space of linear operators $\mathcal{L}(U, V)$. We put for every $t \in [0,T]$

$$\int_0^t \Phi_s dZ(s) = \sum_{j \geq 1} \int_0^t \Phi_s e_j dZ_j(s)$$

where $\int_0^t \Phi_s e_j dZ_j(s)$ is a V valued random variable. Note that the integral exists in $L^2(\Omega, V)$ if

$$\mathbf{E} \left| \int_0^t \Phi_s dZ(s) \right|_V^2 = \sum_j |\|\Phi e_j\|_{\mathcal{H}}|_V^2 < \infty.$$

Remark 5 *If the integrand Φ does not depend on time, then we find*

$$\mathbf{E} \left| \int_0^t \Phi dZ(s) \right|_V^2 = \sum_j |\Phi e_n|_V^2 \, \mathbf{E} \left| \int_0^t dz_j(s) \right|^2 = t^{2H} \sum_j |\Phi e_n|_V^2$$

and it can be seen that the integral $\int_0^t \Phi dZ(s)$ exists if and only if Φ is a Hilbert-Schmidt operator.

Now we introduce stochastic evolution equations driven by the infinite-dimensional Rosenblatt process. Let $A : Dom(A) \subset V \to V$ be the infinitesimal generator of the strongly continuous semigroup $(e^{tA})_{t \in [0,T]}$. We study the equation

$$dX(t) = AX(t)dt + \Phi dZ(t) \tag{23}$$

where $X(0) = x \in V$ and $\Phi \in \mathcal{L}(U; V)$. We will consider *mild solution* of (23), that is, (when it exists), it can be written as

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} \Phi dZ(s). \tag{24}$$

We will not assume that Φ is Hilbert-Schmidt (although the integral $\int \Phi dZ$ exists if and only if Φ is Hilbert-Schmidt); this assumption is unnecessary because, under suitable hypothesis on A, the integral $\int_0^t e^{(t-s)A} \Phi dZ(s)$ will exist even when Φ is not Hilbert-Schmidt operator.

The method used in the fBm case will allow to prove the next theorem.

Theorem 1 *Let Z be given by (22) with $H \in (\frac{1}{2}, 1)$. Consider $\Phi \in \mathcal{L}(U; V)$ and $A : Dom(A) \subset V \to V$ be a negative self-adjoint operator. Then there exists a mild solution X of the equation (23) if and only if the operator $\Phi^\star G_H(-A)\Phi$ is a trace class operator, where*

$$G_H(\lambda) = (\max(\lambda, 1))^{-2H}. \tag{25}$$

Remark 6 *in [44] in the fBm case is assumed that the spectrum of A, $\sigma(A) \subset -(\infty, -l]$ with $l > 0$. The situation when 0 is an accumulation point of the spectrum is not treated; this case is solved in [12].*

Proof: Since

$$\mathbf{E}\left|\int_0^t e^{(t-s)A}\Phi dZ(s)\right|_V^2 = c(H)\sum_n \int_0^t \int_0^t \langle e^{(t-u)A}\Phi e_n, e^{(t-v)A}\Phi e_n\rangle_V |u-v|^{2H-2}dudv$$

and what it follows is a deterministic problem that can be solved as in [44]. ∎

If $\mathbb{S}^1$ denotes the unit circle and A is the Laplacian on the circle, we have

Corollary 2 *Assume that $U = V = L^2(\mathbb{S}^1)$ and $A = \Delta$ is the Laplacian on U. Denote by $(e_n, f_n)_{n\geq 1}$ the eigenvectors of Δ that form an orthonormal basis in $L^2(\mathbb{S}^1)$. Let $(q_n)_n$ be a bounded sequence of non-negative real numbers and*

$$Z(t) = \sum_n \sqrt{q_n}e_n z_n(t) + \sum_n \sqrt{q_n}f_n \tilde{z}_n(t)$$

with $(z_j, \tilde{z}_j)_j$ independent real Rosenblatt processes. Then (23) has an unique mild solution such that $X(t) \in L^2(\Omega, V)$ if an only if

$$\sum_n q_n n^{-4H} < \infty.$$

5   Pathwise stochastic calculus

At this point, we will start to develop a stochastic integration theory with respect to the Rosenblatt process. In general, for processes that are not semimartingales, the Itô's theory cannot be applied. One needs generalized alternative ways to integrate stochastically with respect to such processes. In general these generalized method are essentially of two types: the first is the pathwise type calculus and (here we included the rough path analysis [34] and the stochastic calculus via regularization [36]) and the second type is Malliavin calculus and the Skorohod integration theory [27]. In general the pathwise type calculus is connected to the trajectorial regularity and/or the covariance structure of the integrator process. The Malliavin calculus instead is very related to the Gaussian character of the driven process.

Since the Rosenblatt process with $H > \frac{1}{2}$ has zero quadratic variation (see [37]) and regular paths (Hölder continuous of order $H-\varepsilon$), the pathwise calculus can be naturally applied to construct stochastic integrals with respect to it. Here we choose to use the approach of Russo and Vallois. Let us list first the main ingredients of the stochastic calculus via regularization. Let $(X_t)_{t\geq 0}$ and and $(Y_t)_{t\geq 0}$ continuous processes. We introduce, for every t,

$$I^-(\varepsilon, Y, dX) = \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon}ds, \quad I^+(\varepsilon, Y, dX) = \int_0^t Y_s \frac{X_s - X_{(s-\varepsilon)+}}{\varepsilon}ds,$$

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$$I^0(\varepsilon, Y, dX) = \int_0^t Y_s \frac{X_{s+\varepsilon} - X_{(s-\varepsilon)_+}}{2\varepsilon} ds$$

and

$$C_\varepsilon(X, Y)(t) = \int_0^t \frac{(X_{s+\varepsilon} - X_{(s-\varepsilon)_+})(Y_{s+\varepsilon} - Y_{(s-\varepsilon)_+})}{\varepsilon} ds.$$

Then *the forward, backward and symmetric integrals of Y with respect to X will be given*

$$\int_0^t Y d^- X = \lim_{\varepsilon \to 0^+} I^-(\varepsilon, Y, dX), \quad \int_0^t Y d^+ X = \lim_{\varepsilon \to 0^+} I^+(\varepsilon, Y, dX),$$

and

$$\int_0^t Y d^0 X = \lim_{\varepsilon \to 0^+} I^0(\varepsilon, Y, dX) \tag{26}$$

provided that the above limits exist uniformly in probability (ucp). *The covariation of X and Y is defined as*

$$[X, Y]_t = ucp - \lim_{\varepsilon \to 0^+} C_\varepsilon(X, Y)(t).$$

If $X = Y$ we denote $[X, X] = [X]$ and when $[X]$ exists then X is said to be a finite quadratic variation process. When $[X] = 0$, then X is called a zero quadratic variation process.

The Rosenblatt process is clearly a zero quadratic variation process since

$$\mathbf{E} C_\varepsilon(Z, Z)(t) = \mathbf{E} \int_0^t \frac{1}{\varepsilon} (X_{s+\varepsilon} - X_s)^2 ds = t\varepsilon^{2H-1} \to_{\varepsilon \to 0} 0.$$

Therefore the stochastic calculus via regularization can be directly applied to it. Precisely, it follows from Proposition 4.2 of the Russo and Vallois survey [38] that every $f \in C^2(\mathbb{R})$, the integrals

$$\int_0^t f'(X) d^- X, \quad \int_0^t f'(X) d^+ X, \quad \int_0^t f'(X) d^0 X$$

exist and are equal and we have the Itô's formula

$$f(X_t) = f(X_0) + \int_0^t f'(X) d^0 X. \tag{27}$$

Remark 7 *An immediate consequence of the existence of the quadratic variation of the Rosenblatt process is the existence and uniqueness of the solution of a Stratonovich stochastic differential equation driven by Z. Concretely, if $\sigma : \mathbb{R} \to \mathbb{R}$ and $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy some regularity assumptions and V is a locally bounded variation process, then the equation*

$$dX(t) = \sigma(X(t)) d^0 Z(t) + b(t, X(t)) dV(t) \tag{28}$$

with $X(0) = G$ where G is an arbitrary random variable, has an unique solution (see [37] for the definition of the solution).

6   Skorohod integral with respect to the Rosenblatt process

In this part we define a divergence integral with respect to $(Z(t))_{t \in [0,T]}$. Constructing generalized Skorohod integrals with respect to processes that are not necessarily Gaussian or semimartingales constitutes a frequent topic. For results in this direction, we refer among others, to [33], [19], [26], [17] or [22].

We will need some basic elements of the Malliavin calculus with respect to a Wiener process $(W_t)_{t \in [0,T]}$. By $\mathcal{S}$ we denote the class of smooth random variables of the form

$$F = f(W_{t_1}, \ldots, W_{t_n}), \qquad t_1, \ldots, t_n \in [0, T] \tag{29}$$

where $f \in C_b^\infty(\mathbb{R}^n)$. If F is of the form (29), its Malliavin derivative is defined as

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W_{t_1}, \ldots, W_{t_n}) 1_{[0,t_i]}(t), \qquad t \in [0, T].$$

The operator D is an unbounded closable operator and it can be extended to the closure of $\mathcal{S}$ (denoted $\mathbb{D}^{k,p}$, $k \geq 1$ integer, $p \geq 2$) with respect to the norm

$$\|F\|_{k,p}^p = \mathbf{E}|F|^p + \sum_{j=1}^k \mathbf{E}\|D^{(j)}F\|_{L^2([0,T]^j)}^p, \qquad F \in \mathcal{S}, k \geq 1, p \geq 2$$

where the jth derivative $D^{(j)}$ is defined by iteration.

The Skorohod integral δ is the adjoint of D. Its domain is

$$Dom(\delta) = \{u \in L^2([0,T] \times \Omega) / \left| \mathbf{E} \int_0^T u_s D_s F ds \right| \leq C\|F\|_2\}$$

and D and δ satisfy the duality relationship

$$\mathbf{E}(F\delta(u)) = \mathbf{E} \int_0^T D_s F u_s ds, \qquad F \in \mathcal{S}, u \in Dom(\delta). \tag{30}$$

We define $\mathbb{L}^{k,p} = L^p([0,T]; \mathbb{D}^{k,p})$. Note that $\mathbb{L}^{k,p} \subset Dom(\delta)$. We denote $\delta(u) = \int_0^T u_s \delta W_s$. We will need the integration by parts formula

$$F\delta(u) = \delta(Fu) + \int_0^T D_s F u_s \tag{31}$$

if $F \in \mathbb{D}^{1,2}$ and $u \in \mathbb{L}^{1,2}$.

We also mention that the Skorohod integral with respect to the fBm B^H with Hurst parameter $H > \frac{1}{2}$ is defined through a transfer operator

$$\int_0^T g_s dB_s^H = \int_0^T \int_s^T g_r \frac{\partial K^H}{\partial r}(r,s) dr dW_s \tag{32}$$

where the integral in the right side above is a Skorohod integral with respect to W. moreover g is Skorohod integrable with respect to b^H if the quantity $\int_s^T g_r \frac{\partial K^H}{\partial r}(r, s)dr$ is Skorohod integrable with respect to W.

Definition 1 *Let us consider a square integrable stochastic process $(g_s)_{s \in [0,T]}$. Following (16) and (32) we define its Skorohod integral with respect to Z by*

$$
\int_0^T g_s dZ(s)
$$
$$
:= \int_0^T \int_0^T I(g)(y_1, y_2) dB(y_1) dB(y_2)
$$
$$
= \int_0^T \int_0^T \left(\int_{y_1 \vee y_2}^T g(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right) dB(y_1) dB(y_2). \qquad (33)
$$

We will say that a process g is Skorohod integrable with respect to Z if the process $Ig \in Dom\delta^{(2)}$, where $\delta^{(2)}$ is the double Skorohod integral with respect to the Brownian motion B.

We refer to [28] for the study of double (and multiple) Skorohod integrals.

Remark 8 *Note that the Skorohod integral coincide with the Wiener integral if the integrand g is a deterministic function in $\mathcal{H}$. Another Skorohod integral with respect to Z has been introduced in [22] as the adjoint of some Malliavin derivative with respect to Z but this integral does not coincide with the Wiener integral for deterministic integrands.*

Next lemma gives a condition that ensures the Skorohod integrability.

Lemma 1 *Let $g \in L^2(\Omega; \mathcal{H})$ be such that $g \in \mathbb{L}^{2,2}$ and*

$$
\mathbf{E} \int_0^T \int_0^T \|D_{x_1, x_2} g\|_{\mathcal{H}}^2 dx_1 dx_2 < \infty. \qquad (34)
$$

Then g is Skorohod integrable with respect to Z and

$$
\mathbf{E} \left| \int_0^T g_s \delta Z(s) \right|^2 \leq cst. \left[\mathbf{E}\|g\|_{\mathcal{H}}^2 + \mathbf{E} \int_0^T \int_0^T \|D_{x_1, x_2} g\|_{\mathcal{H}}^2 dx_1 dx_2 \right]. \qquad (35)
$$

Proof: We use Meyer's inequality for the double Skorohod integral (see [28], pag. 320) and

we obtain

$$
\begin{aligned}
\mathbf{E}\left|\int_0^T g_s\delta Z(s)\right|^2 \;\leq\; & cst.\left[\mathbf{E}\int_0^T\int_0^T I(g)(y_1,y_2)^2 dy_1dy_2\right.\\
& \left.+\mathbf{E}\int_0^T\int_0^T\int_0^T\int_0^T \left(D_{x_1,x_2}I(g)(y_1,y_2)\right)^2 dx_1dx_2dy_1dy_2\right]\\
=\; & cst.\left[\mathbf{E}H(2H-1)\int_0^T\int_0^T g(u)g(v)|u-v|^{2H-2}dvdu\right.\\
& \left.+\int_0^T\int_0^T dx_1dx_2\left(\int_0^T\int_0^T D_{x_1,x_2}g(u)D_{x_1,x_2}g(v)|u-v|^{2H-2}dvdu\right)\right]\\
=\; & cst.\left[\mathbf{E}\|g\|_{\mathcal{H}}^2+\mathbf{E}\int_0^T\int_0^T \|D_{x_1,x_2}g\|_{\mathcal{H}}^2 dx_1dx_2\right].
\end{aligned}
$$

$\blacksquare$

Corollary 3 *If $g\in L^2(\Omega;|\mathcal{H}|)$ be such that $g\in\mathbb{L}^{2,2}$ and*

$$
\mathbf{E}\int_0^T\int_0^T \|D_{x_1,x_2}g\|_{|\mathcal{H}|}^2 dx_1dx_2 < \infty. \tag{36}
$$

Then g is Skorohod integrable with respect to Z and

$$
\mathbf{E}\left|\int_0^T g_s\delta Z(s)\right|^2 \leq cst.\|g\|^2 \tag{37}
$$

where

$$
\|g\|^2 = \left[\mathbf{E}\|g\|_{|\mathcal{H}|}^2+\mathbf{E}\int_0^T\int_0^T \|D_{x_1,x_2}g\|_{|\mathcal{H}|}^2 dx_1dx_2\right].
$$

Example 1 *The Rosenblatt process Z is Skorohod integrable with respect to Z and*

$$
\mathbf{E}\left|\int_0^T g_s\delta Z(s)\right|^2 \leq cst.\int_0^T\int_0^T R(u,v)|u-v|^{2H-2}dudv.
$$

Proof: We treat the two terms in the right side of (35). Clearly

$$
\mathbf{E}\|Z\|_{\mathcal{H}}^2 = cst.\int_{.0}^T\int_0^T R(u,v)|u-v|^{2H-2}dudv
$$

We have, for every $x_1,x_2\in[0,T]$,

$$
D_{x_1,x_2}Z(u) = 2d(H)1_{[0,u]^2}(x_1,x_2)\int_{x_1\vee x_2}^u \frac{\partial K^{H'}}{\partial u'}(u',x_1)\frac{\partial K^{H'}}{\partial u'}(u',x_2)du'
$$

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and then (note also that $\frac{\partial K^{H'}}{\partial t}(t,s)$ is positive and therefore we omitt the absolute value at a certain point)

$$
\mathbf{E} \int_0^T \int_0^T \|D_{x_1,x_2} g\|_{|\mathcal{H}|}^2 dx_1 dx_2
$$

$$
= \int_0^T \int_0^T dx_1 dx_2 \int_{x_1 \vee x_2}^T \int_{x_1 \vee x_2}^T |u-v|^{2H-2} du dv
$$

$$
\left| \int_{x_1 \vee x_2}^u \frac{\partial K^{H'}}{\partial u'}(u',x_1) \frac{\partial K^{H'}}{\partial u'}(u',x_2) du' \int_{x_1 \vee x_2}^v \frac{\partial K^{H'}}{\partial v'}(v',x_1) \frac{\partial K^{H'}}{\partial v'}(v',x_2) dv' \right|
$$

$$
= \int_0^T \int_0^T |u-v|^{2H-2} du dv \int_0^u \int_0^v \left(\int_0^{u' \wedge v'} \frac{\partial K^{H'}}{\partial u'}(u',x_1) \frac{\partial K}{\partial v'}(v',x_1) dx_1 \right)^2
$$

$$
= cst. \int_0^T \int_0^T R(u,v)|u-v|^{2H-2} du dv.
$$

∎

We finish the section by a result on the continuity of the indefinite Skorohod integral process. This shows that the indefinite keeps the same order of Hölder regularity as the Rosenblatt process.

Proposition 4 *Let $g \in \mathbb{L}^{2,p}$ such that*

$$
\sup_r \|g_r\|_{2,p} \leq \infty.
$$

Then the indefinite Skorohod integral process $\left(X_t = \int_0^t g_s \delta Z(s), t \in [0,T] \right)$ admits a Hölder continuous version of order $\delta < H$.

Proof: We can write

$$
\begin{aligned}
X_t - X_s &= \int_s^t \int_s^t \left(\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial u}(u,y_2) du \right) dB(y_1) dB(y_2) \\
&\quad + 2 \int_0^s \int_s^t \left(\int_{y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial u}(u,y_2) du \right) dB(y_1) dB(y_2) \\
&\quad + \int_0^s \int_0^s \left(\int_s^t \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial u}(u,y_2) du \right) dB(y_1) dB(y_2) \\
&:= J_1 + 2J_2 + J_3.
\end{aligned}
$$

Then

$$
\mathbf{E} |X_t - X_s|^p \leq c(p) \mathbf{E} \left(J_1^p + J_2^p + J_3^p \right).
$$

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By Meyer's inequality ([28], pag. 320)

$$
\begin{aligned}
\mathbf{E}|J_1|^p &\leq c(p) \left| \int_s^t \int_s^t \mathbf{E} \left(\int_{y_1 \vee y_2}^t g_u \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right)^2 dy_1 dy_2 \right|^{\frac{p}{2}} \\
&\quad + c(p)\mathbf{E} \left| \int_s^t \int_s^t \int_0^t \int_0^t \left[D_{x_1, x_2} \int_{y_1 \vee y_2}^t g_u \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right]^2 dx_1 dx_2 dy_1 dy_2 \right|^{\frac{p}{2}} \\
&= c(p, H) \left| \mathbf{E} \int_s^t \int_s^t |g(u)g(v)||u - v|^{2H-2} \right|^{\frac{p}{2}} \\
&\quad + c(p, H)\mathbf{E} \left| \int_0^t \int_0^t dx_1 dx_2 \int_s^t \int_s^t |D_{x_1, x_2} g_u D_{x_1, x_2} g_v||u - v|^{2H-2} dv du \right|^{\frac{p}{2}} \\
&\leq c(p, H) \sup_r \|g_r\|_{2,p}^p \left| \int_s^t \int_s^t |u - v|^{2H-2} dv du \right|^{\frac{p}{2}} \\
&= c(p, H) \sup_r \|g_r\|_{2,p}^p (t - s)^{pH}.
\end{aligned}
$$

In a similar way, we can find the same bound for the terms J_2 and J_3 (see also [4], proof of Proposition 1). The conclusion will following by the Kolmogorov's continuity criterium. ■

7 The relation between the pathwise and the Skorohod integrals

Let g a stochastic process. Recall that its forward integral with respect to Z is the limit ucp as $\varepsilon \to 0$ of

$$
I^-(\varepsilon, g, dZ) = \frac{1}{\varepsilon} \int_0^T g_s(Z(s + \varepsilon) - Z(s)) ds = \frac{1}{\varepsilon} \int_0^T g_s \delta^{(2)} \left(f_{s+\varepsilon}(\cdot, *) - f_s(\cdot, *) \right) ds \qquad (38)
$$

where the kernel f_s is given by

$$
f_s(x, y) = d(H) 1_{[0,s]^2}(x, y) \int_{x \vee y}^s \frac{\partial K^{H'}}{\partial u}(u, x) \frac{\partial K^{H'}}{\partial u}(u, y) du. \qquad (39)
$$

We will need a formula by [28]: if $F \in \mathbb{D}^{2,2}$, $u \in L^2([0,T]^2 \times \Omega)$ such that for every s, $u(\cdot, s) \in Dom(\delta)$, then $Fu \in Dom(\delta^{(2)})$ and

$$
F\delta^{(2)}(u) = \delta^{(2)}(Fu) + 2 \int_0^T D_\alpha F \delta(u(\cdot, \alpha)) d\alpha - \int_0^T \int_0^T D_{\alpha, \beta}^{(2)} Fu(\alpha, \beta) d\alpha d\beta. \qquad (40)
$$

We apply relation (40) to (38) and we obtain

$$
\begin{aligned}
I^-(\varepsilon, g, dZ) \;=\; & \frac{1}{\varepsilon} \int_0^T \delta^{(2)} \left(g_s \left(f_{s+\varepsilon}(\cdot, *) - f_s(\cdot, *) \right) \right) ds \\
& + \frac{2}{\varepsilon} \int_0^T \int_0^T D_\alpha g_s \delta \left(f_{s+\varepsilon}(\cdot, \alpha) - f_s(\cdot, \alpha) \right) d\alpha ds \\
& - \frac{1}{\varepsilon} \int_0^T \int_0^T \int_0^T D^{(2)}_{\alpha, \beta} g_s \left(f_{s+\varepsilon}(\beta, \alpha) - f_s(\beta, \alpha) \right) d\beta d\alpha ds.
\end{aligned}
\tag{41}
$$

We can already observe, besides the first divergence type term, the appearance of two trace terms. Recall that in the fBm case the corresponding term $I^-(\varepsilon, g, dB^H)$ can be decomposed in a divergence term plus a only a trace term.

Definition 2 *We say that a stochastic process $g \in \mathbb{L}^{1,2}$ admits a trace of order 1 if*

$$
\frac{1}{\varepsilon} \int_0^T \int_0^T D_\alpha g_s \delta \left(f_{s+\varepsilon}(\cdot, \alpha) - f_s(\cdot, \alpha) \right) d\alpha ds
\tag{42}
$$

converges in probability as $\varepsilon \to 0$. The limit will be denoted by $Tr^{(1)}(D^{(1)}g)$.

We say that a stochastic process $g \in \mathbb{L}^{2,2}$ admits a trace of order 2 if

$$
\frac{1}{\varepsilon} \int_0^T \int_0^T \int_0^T D^{(2)}_{\alpha, \beta} g_s \left(f_{s+\varepsilon}(\beta, \alpha) - f_s(\beta, \alpha) \right) d\beta d\alpha ds
\tag{43}
$$

converges in probability as $\varepsilon \to 0$. The limit will be denoted by $Tr^{(2)}(D^{(2)}g)$.

We have the following relation between the divergence and the pathwise integral.

Theorem 2 *Let $g \in \mathbb{L}^{2,2}$ such that*

$$
\mathbf{E}\|g\|^2_{|\mathcal{H}|} + \mathbf{E} \int_0^T \int_0^T \|D_{x_1, x_2} g\|^2_{|\mathcal{H}|} dx_1 dx_2 < \infty.
$$

Assume that g has traces of order 1 and 2. Then g is forward integrable with respect to Z and it holds

$$
\int_0^T g_s d^- Z(s) = \int_0^T g_s \delta Z(s) + 2Tr^{(1)}(D^{(1)}g) - Tr^{(2)}(D^{(2)}g).
\tag{44}
$$

Proof: By (41) and Definition 2, it suffices to show that the term

$$
A_\epsilon = \frac{1}{\varepsilon} \int_0^T \delta^{(2)} \left(g_s \left(f_{s+\varepsilon}(\cdot, *) - f_s(\cdot, *) \right) \right) ds
$$

converges to

$$
\int_0^T g_s \delta Z(s)
$$

in $L^2(\Omega)$ as $\varepsilon \to 0$.

We can write, by Fubini,

$$
\begin{aligned}
A_\varepsilon &= \frac{1}{\varepsilon} \int_0^T ds \int_0^T \int_0^T g_s \left(f_{s+\varepsilon}(y_1, y_2) - f_s(y_1, y_2) \right) dB(y_1) dB(y_2) \\
&= \int_0^T \int_0^T dB(y_1) dB(y_2) \int_{y_1 \vee y_2}^T g^\varepsilon(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \\
&= \int_0^T \int_0^T I(g^\varepsilon)(y_1, y_2) dB(y_1) dB(y_2) = \int_0^T g_s^\varepsilon \delta Z(s)
\end{aligned}
$$

where we denoted by

$$
g^\varepsilon(u) = \frac{1}{\varepsilon} \int_{u-\varepsilon}^u g_s ds. \tag{45}
$$

By using (37), it is sufficient to check that

$$
g^\varepsilon \to_{\varepsilon \to 0} g \text{ in } L^2(\Omega; \mathcal{H})
$$

and

$$
\int_0^T \int_0^T \mathbf{E} \| D_{x_1, x_2}(g^\varepsilon - g) \|_{\mathcal{H}}^2 dx_1 dx_2 \to_{\varepsilon \to 0} 0.
$$

We will show that

$$
\| g^\varepsilon \|_{|\mathcal{H}|} \leq c(H) \| g \|_{|\mathcal{H}|} \tag{46}
$$

and

$$
\int_0^T \int_0^T \| D_{x_1, x_2} g^\varepsilon \|_{|\mathcal{H}|}^2 dx_1 dx_2 \leq c(H) \int_0^T \int_0^T \| D_{x_1, x_2} g \|_{|\mathcal{H}|}^2 dx_1 dx_2. \tag{47}
$$

The bound (46) has been proved in [5], proof of Proposition 3, Step 1. Concerning the bound (47), we can write

$$
\begin{aligned}
&\int_0^T \int_0^T \| D_{x_1, x_2} g^\varepsilon \|_{|\mathcal{H}|}^2 dx_1 dx_2 \\
&= c(H) \int_0^T \int_0^T dx_1 dx_2 \int_0^T \int_0^T |D_{x_1, x_2} g_u^\varepsilon| |D_{x_1, x_2} g_v^\varepsilon| |u - v|^{2H-2} du dv \\
&\leq c(H) \frac{1}{\varepsilon^2} \int_0^T \int_0^T dx_1 dx_2 \int_0^T \int_0^T du dv |u - v|^{2H-2} \int_{v-\varepsilon}^v \int_{u-\varepsilon}^u ds ds' |D_{x_1, x_2} g_s D_{x_1, x_2} g_{s'}| \\
&\leq c(H) \int_0^T \int_0^T dx_1 dx_2 \int_0^T \int_0^T ds ds' |D_{x_1, x_2} g_s D_{x_1, x_2} g_{s'}| \left(\frac{1}{\varepsilon^2} \int_s^{s+\varepsilon} \int_{s'}^{s'+\varepsilon} |u - v|^{2H-2} du dv \right).
\end{aligned}
$$

It follows from [5], proof of Proposition 3, Step 1, that

$$
\frac{1}{\varepsilon^2} \int_s^{s+\varepsilon} \int_{s'}^{s'+\varepsilon} |u - v|^{2H-2} du dv \leq c(H) |s - s'|^{2H-2}
$$

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and thus (47) follows.

Now we can finish the proof proceeding as in [5], proof of Proposition 3, Step 3. Consider a sequence g^n of simple processes of the form $g^n = \sum_{i=0}^{n-1} F_i 1_{(t_i, t_{i+1}]}$ with $F_i \in \mathcal{S}$ and $t_i \in [0, T]$ such that $\|g_n - g\| \to 0$ in $L^2(\Omega)$ when $n \to \infty$ (the existence of a such sequence follows easily by the densite of $\mathcal{E}$ in $\mathcal{H}$). Then by (37) we have that

$$\int_0^T g_s^n \delta Z(s) \to_{n \to \infty} \int_0^T g_s \delta Z(s)$$

Denote by $g^{n,\varepsilon}$ the approximation process of the form (45) associated to g^n. We can write, for any $\varepsilon > 0$ and $n \geq 1$,

$$\mathbf{E} \left| \int_0^T g_s^\varepsilon \delta Z(s) - \int_0^T g_s \delta Z(s) \right|^2 \leq 3 \left(\mathbf{E} \left| \int_0^T g_s^\varepsilon \delta Z(s) - \int_0^T g_s^{n,\varepsilon} \delta Z(s) \right|^2 \right.$$

$$\left. + \mathbf{E} \left| \int_0^T g_s^{n,\varepsilon} \delta Z(s) - \int_0^T g_s^n \delta Z(s) \right|^2 + \mathbf{E} \left| \int_0^T g_s^n \delta Z(s) - \int_0^T g_s \delta Z(s) \right|^2 \right).$$

By (46) and (47) it follows that for n large enough and for any $\delta > 0$

$$\mathbf{E} \left| \int_0^T g_s^\varepsilon \delta Z(s) - \int_0^T g_s \delta Z(s) \right|^2 \leq 3 \left(\mathbf{E} \left| \int_0^T g_s^{n,\varepsilon} \delta Z(s) - \int_0^T g_s^n \delta Z(s) \right|^2 + \delta \right)$$

and we can conclude by taking $\varepsilon \to 0$. ∎

8   On the Itô formula in the Skorohod sense

We study Itô's formula for the Rosenblatt process in the divergence sense. As we mentioned before, the Gaussian nature of the integrator process is essential in the framework of the divergence calculus and this fact can be entirely observed here. We are actually able to prove Skorohod Itô's formula only in two particular cases; but more relevant than these formulas, which are not easily tractable, is the fact that one can observe from the computations contained here that the standard method to obtain divergence type change of variables formulas (see e.g. [27]) does not work here, in the sense that one cannot hope to obtain Itô's formulas that stop at f''.

We will deduce the Skorohod Itô formula by using the pathwise Itô formula. Recall that for any function $f \in C^2(\mathbb{R})$

$$\begin{aligned} f(Z(t)) &= f(0) + \int_0^t f'(Z(s)) d^- Z(s) \\ &= f(0) + \int_0^t f'(Z(s)) \delta Z(s) + 2Tr^{(1)}(D^{(1)} f'(Z(s))) - Tr^{(2)}(D^{(2)} f''(Z(s))) \end{aligned}$$

provided that the above terms exist.

8.1 The trace of order 1

Recall that

$$Tr^{(1)}(D^{(1)}f'(Z(s))) = ucp - \lim_{\varepsilon \to 0} B_\varepsilon$$

where

$$
\begin{aligned}
B_\varepsilon &= \frac{1}{\varepsilon} \int_0^t ds \int_0^t d\alpha D_\alpha f'(Z(s))\delta\left(f_{s+\varepsilon}(\cdot,\alpha) - s_s(\cdot,\alpha)\right) \\
&= \frac{1}{\varepsilon} \int_0^t ds \int_0^t d\alpha f''(Z(s))D_\alpha Z(s)\delta\left(f_{s+\varepsilon}(\cdot,\alpha) - f_s(\cdot,\alpha)\right).
\end{aligned}
$$

The Malliavin derivative of $Z(s)$ is given by

$$D_\alpha Z(s) = 2d(H)1_{[0,s]}(\alpha) \left(\int_0^s \left(\int_{\alpha \vee y_1}^s \frac{\partial K^{H'}}{\partial u}(u,\alpha)\frac{\partial K^{H'}}{\partial u}(u,y_1)du \right) dB(y_1) \right) \tag{48}$$

Thus

$$
\begin{aligned}
B_\varepsilon &= \frac{2}{\varepsilon}d(H) \int_0^T ds f''(Z(s)) \int_0^s \delta\left(f_{s+\varepsilon}(\cdot,\alpha) - f_s(\cdot,\alpha)\right) d\alpha \\
&\quad \int_0^s \left(\int_{\alpha \vee y_1}^s \frac{\partial K^{H'}}{\partial u}(u,\alpha)\frac{\partial K^{H'}}{\partial u}(u,y_1)du \right) dB(y_1)
\end{aligned}
$$

where f_s is given by (39). By using the integration by parts formula (31) it holds that

$$
\begin{aligned}
B_\varepsilon &= \frac{2}{\varepsilon}d(H) \int_0^T ds f''(Z(s)) \int_0^s d\alpha \\
&\quad \int_0^s \left[\delta\left(f_{s+\varepsilon}(\cdot,\alpha) - f_s(\cdot,\alpha)\right) \int_{\alpha \vee y_1}^s \frac{\partial K^{H'}}{\partial u}(u,\alpha)\frac{\partial K^{H'}}{\partial u}(u,y_1)du \right] dB(y_1) \\
&\quad + \frac{2}{\varepsilon}d(H) \int_0^T ds f''(Z(s)) \int_0^s d\alpha \\
&\quad \int_0^s \left[\int_{\alpha \vee y_1}^s \frac{\partial K^{H'}}{\partial u}(u,\alpha)\frac{\partial K^{H'}}{\partial u}(u,y_1)du(f_{s+\varepsilon}(y_1,\alpha) - f_s(y_1,\alpha) \right] dy_1 \\
&:= B_\varepsilon^1 + B_\varepsilon^2.
\end{aligned}
$$

We regard first the term B_ε^2 because it can be treated in the same manner for any function f. We can write

$$
\begin{aligned}
B_\varepsilon^2 &= \frac{2}{\varepsilon} d(H)^2 \int_0^T ds\, f''(Z(s)) \int_0^s d\alpha \\
&\quad \int_0^s dy_1 \int_{\alpha \vee y_1}^s \frac{\partial K^{H'}}{\partial u}(u,\alpha) \frac{\partial K^{H'}}{\partial u}(u,y_1) du \\
&\quad \left[1_{[0,s+\varepsilon]^2}(y_1,\alpha) \int_{\alpha \vee y_1}^{s+\varepsilon} \frac{\partial K^{H'}}{\partial v}(v,\alpha) \frac{\partial K^{H'}}{\partial v}(v,y_1) dv \right. \\
&\quad \left. - 1_{[0,s]^2}(y_1,\alpha) \int_{\alpha \vee y_1}^s \frac{\partial K^{H'}}{\partial v}(v,\alpha) \frac{\partial K^{H'}}{\partial v}(v,y_1) dv \right] \\
&= \frac{2}{\varepsilon} d(H)^2 \int_0^T ds\, f''(Z(s)) \int_0^s du \int_s^{s+\varepsilon} dv \left(\int_0^{u \wedge v} \frac{\partial K^{H'}}{\partial u}(u,\alpha) \frac{\partial K^{H'}}{\partial v}(v,\alpha) d\alpha \right)^2 \\
&= 2A(H)^2 \int_0^t dv \int_0^v du |u-v|^{2H-2} \frac{1}{\varepsilon} \int_{(v-\varepsilon)\vee u}^v f''(Z(s)) ds
\end{aligned}
$$

with $A(H) = H'(2H'-1)d(H)$ and we have

$$
\begin{aligned}
B_\varepsilon^2 &= 2A(H)^2 \int_0^t dv \int_0^v du |u-v|^{2H-2} \left(\frac{1}{\varepsilon} \int_{(v-\varepsilon)}^v f''(Z(s)) ds \right) \\
&\quad + 2A(H)^2 \int_0^t dv \int_{v-\varepsilon}^v du |u-v|^{2H-2} \left(\frac{1}{\varepsilon} \int_u^v f''(Z(s)) ds \right). \quad (49)
\end{aligned}
$$

Therefore we have the convergence in $L^1(\Omega)$ as $\varepsilon \to 0$

$$
B_\varepsilon^2 \to 2A(H)^2 \int_0^T \int_0^u f''(Z(v))|u-v|^{2H-2} dv\, du = \frac{A(H)^2}{2H-1} \int_0^T \int_0^T f''(Z(u))u^{2H-1} du \quad (50)
$$

since the first summand in (49) converges to the limit and the second one goes to zero by the dominated convergence theorem.

The study of the term B^1 is rather difficult to be done in general. We will restrict ourselves to its study in some particular cases.

The case $f(x) = x^2$:
We have

$$
\begin{aligned}
B_\varepsilon^1 &= \frac{4}{\varepsilon} d(H) \int_0^t ds \int_0^s d\alpha \int_0^t \int_0^t dB(y_1) dB(y_2) \\
&\quad \int_{\alpha \vee y_1}^s \frac{\partial K^{H'}}{\partial u}(u,\alpha) \frac{\partial K^{H'}}{\partial u}(u,y_1) du (f_{s+\varepsilon}(y_2,\alpha) - f_s(y_2,\alpha))
\end{aligned}
$$

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and by Fubini we get

$$
\begin{aligned}
B_\varepsilon^1 &= 4d(H)^2 \int_0^t \int_0^t dB(y_1)dB(y_2) \int_{y_2}^t dv \int_{y_1}^v du \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2) \\
&\quad \left(\frac{1}{\varepsilon}\int_{(v-\varepsilon)\vee u}^v ds\right)\left(\int_0^{u\wedge v}\frac{\partial K^{H'}}{\partial u}(u,\alpha)\frac{\partial K^{H'}}{\partial v}(v,\alpha)d\alpha\right) \\
&= 4d(H)^2 H'(2H'-1)\int_0^t\int_0^t dB(y_1)dB(y_2) \\
&\quad \int_{y_2}^t dv \int_{y_1}^v du \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)|u-v|^{2H'-2}\left(\frac{1}{\varepsilon}\int_{(v-\varepsilon)\vee u}^v ds\right)
\end{aligned}
$$

and then one can prove that

$$
\begin{aligned}
B_\varepsilon^1 &\to 4d(H)^2 H'(2H'-1)\int_0^t\int_0^t dB(y_1)dB(y_2) \\
&\quad \int_{y_2}^t dv \int_{y_1}^v du \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)|u-v|^{2H'-2}.
\end{aligned} \tag{51}
$$

The case $f(x)=x^3$: It holds by calculating first the integral $d\alpha$

$$
\begin{aligned}
B_\varepsilon^1 &= \frac{12}{\varepsilon}d(H)^2 H'(2H'-1)\int_0^t Z(s)ds \\
&\quad \left[\int_0^s\int_0^{s+\varepsilon} dB(y_1)dB(y_2)\int_{y_1}^s du\int_{y_2}^{s+\varepsilon}dv\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2}\right. \\
&\quad \left.-\int_0^s\int_0^s dB(y_1)dB(y_2)\int_{y_1}^s du\int_{y_2}^s dv\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2}\right]
\end{aligned}
$$

and the integration by parts formula for the double Skorohod integral gives

$$
B_\varepsilon^1 = B_\varepsilon^{1,1} + B_\varepsilon^{1,2} + B_\varepsilon^{1,3}
$$

where

$$
\begin{aligned}
B_\varepsilon^{1,1} &= \frac{12}{\varepsilon}d(H)^2 H'(2H'-1)\int_0^t ds \\
&\quad \left[\int_0^s\int_0^{s+\varepsilon} dB(y_1)dB(y_2)Z(s)\int_{y_1}^s du\int_{y_2}^{s+\varepsilon}dv\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2}\right. \\
&\quad \left.-\int_0^s\int_0^s dB(y_1)dB(y_2)Z(s)\int_{y_1}^s du\int_{y_2}^s dv\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2}\right],
\end{aligned}
$$

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$$B_\varepsilon^{1,2} = -\frac{48}{\varepsilon} d(H)^3 H'(2H'-1) \int_0^t ds \int_0^s d\alpha$$

$$\int_0^s \left(\int_{\alpha \vee y_1}^s du' \frac{\partial K^{H'}}{\partial u'}(u',\alpha) \frac{\partial K^{H'}}{\partial u'}(u',y_1) \right) dB(y_1)$$

$$\left[\int_0^{s+\varepsilon} dB(y_2) \int_\alpha^s du \int_{y_2}^{s+\varepsilon} dv \frac{\partial K^{H'}}{\partial u}(u,\alpha) \frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2} \right.$$

$$\left. - \int_0^s dB(y_2) \int_\alpha^s du \int_{y_2}^s dv \frac{\partial K^{H'}}{\partial u}(u,\alpha) \frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2} \right]$$

and

$$B_\varepsilon^{1,3} = \frac{24}{\varepsilon} d(H)^3 H'(2H'-1) \int_0^t ds$$

$$\left[\int_0^s \int_0^{s+\varepsilon} dy_1 dy_2 \int_{y_1 \vee y_2}^s \frac{\partial K^{H'}}{\partial u'}(u',y_1) \frac{\partial K^{H'}}{\partial u'}(u',y_2) du' \right.$$

$$\int_{y_1}^s du \int_{y_2}^{s+\varepsilon} dv \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2}$$

$$- \int_0^s \int_0^s dy_1 dy_2 \int_{y_1 \vee y_2}^s \frac{\partial K^{H'}}{\partial u'}(u',y_1) \frac{\partial K^{H'}}{\partial u'}(u',y_2) du'$$

$$\left. \int_{y_1}^s du \int_{y_2}^s dv \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2} \right]$$

Now,

$$B_\varepsilon^{1,1} = 12 d(H)^2 H'(2H'-1) \int_0^t \int_0^t dB(y_1) dB(y_2)$$

$$\int_{y_2}^t dv \int_{y_1}^v du \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2} \left(\frac{1}{\varepsilon} \int_{(v-\varepsilon)\vee u}^v Z(s)ds \right)$$

and we have the convergence as in the proof of Theorem 2

$$B_\varepsilon^{1,1} \to 12 d(H)^2 H'(2H'-1) \int_0^t \int_0^t dB(y_1) dB(y_2)$$

$$\int_{y_2}^t dv \int_{y_1}^v du Z_v \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial v}(v,y_2)|u-v|^{2H'-2} \tag{52}$$

To treat the term $B_\varepsilon^{1,2}$ one needs to use again the integration by parts formula (31) and one

28

obtains

$$
\begin{aligned}
B_\varepsilon^{1,2} \quad \to \quad & -48 d(H)^3 (H'(2H'-1))^2 \int_0^t \int_0^t dB(y_1) dB(y_2) \\
& \int_{y_2}^t dv \int_{y_1}^v du' \int_{y_2}^v du |u-u'|^{2H'-2}|u-v|^{2H'-2}\frac{\partial K^{H'}}{\partial u'}(u',y_1)\frac{\partial K^{H'}}{\partial v}(v,y_2) \\
& -48 d(H)^3 (H'(2H'-1))^3 \int_0^t dv \int_0^v \int_0^v du' du |u-v|^{2H'-2}|u'-v|^{2H'-2}|u-u'|^{2H'-2}.
\end{aligned}
$$

Concerning $B_\varepsilon^{1,3}$ we similarly have

$$
B_\varepsilon^{1,3} \to 24 d(H)^3 (H'(2H'-1))^3 \int_0^t dv \int_0^v \int_0^v du' du |u-v|^{2H'-2}|u'-v|^{2H'-2}|u-u'|^{2H'-2}. \tag{53}
$$

8.2 The trace of order 2

Recall that

$$
Tr^{(2)}\left(D^{(2)}f'(Z(s))\right) = ucp - \lim_{\varepsilon \to 0} C_\varepsilon
$$

where

$$
\begin{aligned}
C_\varepsilon \quad &= \quad \frac{1}{\varepsilon}\int_0^t ds \int_0^t \int_0^t D_{\alpha,\beta}f'(Z(s))\left(f_{s+\varepsilon}(\alpha,\beta) - f_s(\alpha,\beta)\right)d\alpha d\beta \\
&= \quad \frac{1}{\varepsilon}\int_0^t ds \int_0^t \int_0^t \left(f_{s+\varepsilon}(\alpha,\beta) - f_s(\alpha,\beta)\right)d\alpha d\beta \\
&\qquad \left[f''(Z(s))D_{\alpha,\beta}Z(s) + f'''(Z(s))D_\alpha Z(s)D_\beta Z(s)\right] \\
&:= \quad C_\varepsilon^1 + C_\varepsilon^2.
\end{aligned}
$$

We can write

$$
\begin{aligned}
C_\varepsilon \quad &= \quad \frac{2}{\varepsilon}d(H)^2 \int_0^t f''(Z(s))ds \int_0^s du \int_s^{s+\varepsilon} dv \left(\int_0^{u\wedge v}\frac{\partial K^{H'}}{\partial u}(u,\alpha)\frac{\partial K^{H'}}{\partial v}(v,\alpha)d\alpha\right)^2 \\
&= \quad \frac{2}{\varepsilon}d(H)^2(H'(2H'-1))^2 \int_0^t f''(Z(s))ds \int_0^s du \int_s^{s+\varepsilon} dv |u-v|^{2H-2} \\
&= \quad 2 d(H)^2(H'(2H'-1))^2 \int_0^t dv \int_0^v du |u-v|^{2H-2}\left(\frac{1}{\varepsilon}\int_{(v-\varepsilon)\vee u}^v f''(Z(s))ds\right)
\end{aligned}
$$

and then clearly

$$
C_\varepsilon^1 \to \frac{d(H)^2(H'(2H'-1))^2}{2H-1} \int_0^t f''(Z(v))v^{2H-1}dv. \tag{54}
$$

in $L^1(\Omega)$ as $\varepsilon \to 0$.

The term denoted by C_ε^2 can be handled in the following way:

$$
\begin{aligned}
C_\varepsilon^2 &= \frac{1}{\varepsilon}\int_0^t ds\, f'''(Z(s))\int_0^t\int_0^t D_\alpha Z(s)D_\beta Z(s)\,(f_{s+\varepsilon}(\alpha,\beta)-f_s(\alpha,\beta))\,d\alpha d\beta \\
&= \frac{4}{\varepsilon}d(H)^2\int_0^t ds\, f'''(Z(s))\int_0^s\int_0^s d\alpha d\beta\,(f_{s+\varepsilon}(\alpha,\beta)-f_s(\alpha,\beta)) \\
&\quad \left(\int_0^t dB(y_1)\int_{\alpha\vee y_1}^s \frac{\partial K^{H'}}{\partial u}(u,\alpha)\frac{\partial K^{H'}}{\partial u}(u,y_1)du\right)\left(\int_0^t dB(y_2)\int_{\beta\vee y_2}^s \frac{\partial K^{H'}}{\partial v}(v,\beta)\frac{\partial K^{H'}}{\partial v}(v,y_2)dv\right)
\end{aligned}
$$

The case $f(x)=x^3$. In this case

$$
\begin{aligned}
C_\varepsilon^2 &= \frac{24}{\varepsilon}d(H)^2\int_0^t ds\int_0^s\int_0^s d\alpha d\beta\,(f_{s+\varepsilon}(\alpha,\beta)-f_s(\alpha,\beta)) \\
&\quad \int_0^t\int_0^t dB(y_1)dB(y_2))\int_{\alpha\vee y_1}^s \frac{\partial K^{H'}}{\partial u}(u,\alpha)\frac{\partial K^{H'}}{\partial u}(u,y_1)du\int_{\beta\vee y_2}^s \frac{\partial K^{H'}}{\partial v}(v,\beta)\frac{\partial K^{H'}}{\partial v}(v,y_2)dv \\
&\quad +\frac{24}{\varepsilon}\int_0^t ds\int_0^s\int_0^s d\alpha d\beta\,(f_{s+\varepsilon}(\alpha,\beta)-f_s(\alpha,\beta)) \\
&\quad \int_0^s dy_1\int_{\alpha\vee y_1}^s \frac{\partial K}{\partial u}(u,\alpha)\frac{\partial K^{H'}}{\partial u}(u,y_1)du\int_{\beta\vee y_2}^s \frac{\partial K^{H'}}{\partial v}(v,\beta)\frac{\partial K^{H'}}{\partial v}(v,y_2)dv
\end{aligned}
$$

and by the same type of calculations as above we can prove that as $\varepsilon\to 0$

$$
\begin{aligned}
C_\varepsilon^2 &\to 24d(H)^3(H'(2H'-1))^2\int_0^t\int_0^t dB(y_1)dB(y_2))\int_{y_1}^t du'\int_{y_1}^{u'}du\int_{y_2}^{u'}dv \tag{55}\\
&\quad |u-u'|^{2H'-2}|v-u'|^{2H'-2}\frac{\partial K}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial v}(v,y_2) \\
&\quad +24d(H)^3(H'(2H'-1))^3\int_0^t du'\int_0^{u'}\int_0^{u'}dudv \\
&\quad |u-u'|^{2H'-2}|v-u'|^{2H'-2}|u-v|^{2H'-2}. \tag{56}
\end{aligned}
$$

We can summarize

Theorem 3 *We have*

$$
\begin{aligned}
Z(t)^2 &= 2\int_0^t Z(s)\delta Z(s)+t^{2H} \\
&\quad +\frac{4(2H-1)}{H+1}\int_0^t\int_0^t dB(y_1)dB(y_2)\int_{y_2}^t dv\int_{y_1}^u du\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)|u-v|^{2H'-2}.
\end{aligned}
$$

30

and

$$Z(t)^3 = 3 \int_0^t Z(s)^2 \delta Z(s) + 24 \frac{(d(H)H'(2H'-1))^2}{2H-1} \int_0^t Z(s) s^{2H-1} ds$$

$$+24 d(H)^2 H'(2H'-1) \int_0^t \int_0^t dB(y_1) dB(y_2)$$

$$\int_{y_2}^t dv \int_{y_1}^v du Z_v \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) |u-v|^{2H'-2}$$

$$+72 d(H)^3 (H'(2H'-1))^2 \int_0^t \int_0^t dB(y_1) dB(y_2)$$

$$\int_{y_2}^t dv \int_{y_1}^v du' \int_{y_1}^v du |u-u'|^{2H'-2} |u-v|^{2H'-2} \frac{\partial K^{H'}}{\partial u'}(u', y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2)$$

$$+24 d(H)^3 (H'(2H'-1))^3 \int_0^t dv \int_0^v \int_0^v du' du |u-v|^{2H'-2} |u'-v|^{2H'-2} |u-u'|^{2H'-2}.$$

Remark 9 *One can note the appearance of a term involving f''' in the expression of the summand C_ε. Therefore one cannot hope to have Itô's formulas that end with a second derivative term.*

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